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Quantum MV-Algebras and Commutativity

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We introduce a notion of commutativity in quantum MV-algebras (QMV-algebras) and we investigate the corresponding structure of the center. It turns out that the center of a QMV-algebra is a multivalued algebra (MV-algebra). Finally, we prove that the center of the QMV-algebra of all effects on a Hilbert space is irreducible.

1. INTRODUCTION

Quantum MV-algebras (QMV-algebras) were introduced in Giunti (1996) as a non-lattice-theoretic generalization of *MV-algebras* (multivalued algebras; Chang, 1957, 1958) and as an algebraic generalization of *effects* on a Hilbert space. An effect is a positive (and therefore self-adjoint) linear operator dominated by the identity on a Hilbert space $\mathcal{H}$. The spectrum of an effect is contained in the real interval [0,1]. In the unsharp approach to quantum mechanics, effects can be considered as the mathematical representatives of "unsharp properties" of a quantum physical system in that their possible values are contained in [0,1]. In the standard approach to quantum mechanics instead, the mathematical interpretation of the notion of property is given by *projections* on $\mathcal{H}$. Differently from effects, the spectrum of any projection is contained in the two-element set $\{0, 1\}$. The class of all projections determines an *orthomodular lattice* (Kalmbach, 1983), whereas the class of all effects determines a QMV-algebra (Giuntini, 1995b) which is not a lattice.

Both QMV and MV-algebras allow us to to define two operations ($\odot$, $\Cap$), which can be interpreted, from a logical point of view, as the connective "and." Differently from MV-algebras, in QMV-algebras the operation $\Cap$ is

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not generally commutative. MV-algebras are precisely those QMV-algebras where $\Cap$ is commutative. Therefore it seems to be interesting to define an appropriate relation of *commutativity* (and a corresponding notion of *center*) for QMV-algebras, which turns out to be universal in the MV-algebra case. As we will see in Section 3, such a relation generalizes the commutativity (or compatibility) relation definable in every orthomodular lattice.

2. BASIC PROPERTIES OF QMV-ALGEBRAS

We assume that the reader is familiar with Giuntini (1996), although, for convenience, we present most of the pertinent basic definitions and properties of QMV-algebras.

Definition 2.1. A *quantum MV-algebra* (*QMV-algebra*) is a structure $\mathcal{M} = (M, \oplus; {}^*, \mathbf{1}, \mathbf{0})$, consisting of a nonempty set M, two special elements $\mathbf{1}$, $\mathbf{0}$ of M, a binary operation $\oplus$ on M, and a 1-ary operation * on M. The following axioms are required to hold $\forall a,b \in M$ (where $a \odot b := (a^* \oplus b^*)^*$, $a \Cap b := (a \oplus b^*) \odot a$, $a \Cup b := (a \odot b^*) \oplus b$):

(QMV1) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
(QMV2) $a \oplus b = b \oplus a$
(QMV3) $a \oplus a^* = \mathbf{1}$
(QMV4) $a \oplus \mathbf{0} = a$
(QMV5) $a \oplus \mathbf{1} = \mathbf{1}$
(QMV6) $a^{**} = a$
(QMV7) $a \Cup (b \Cap a) = a$
(QMV8) $(a \Cap b) \Cap c = (a \Cap b) \Cap (b \Cap c)$
(QMV9) $a \oplus (b \Cap (a \oplus c)^*) = (a \oplus b) \Cap (a \oplus (a \oplus c)^*)$
(QMV10) $a \oplus (a^* \Cap b) = a \oplus b$
(QMV11) $(a^* \oplus b) \Cup (b^* \oplus a) = \mathbf{1}$

We assume $\odot$ to be more binding than $\oplus$.

Definition 2.2. An *MV-algebra* is a QMV-algebra $\mathcal{M}$ s.t. $\forall a, b \in M$

$$a \Cap b = b \Cap a$$

Lemma 2.1. Let $\mathcal{M}$ be a QMV-algebra. The following conditions are equivalent:

(i) $\mathcal{M}$ is an MV-algebra.
(ii) $\forall a,b \in M$: If $a^* \oplus b = \mathbf{1}$, then $a \preccurlyeq b$.

Example 2.1 (Standard QMV-algebra). Let $E(\mathcal{H})$be the class of all effects of a Hilbert space $\mathcal{H}$. $E(\mathcal{H})$ coincides with the class of all bounded linear operators between $\mathbb{0}$ and $\mathbb{1}$, where $\mathbb{0}$ and $\mathbb{1}$ are the null and the identity operator, respectively. The operations $\oplus$ and $*$ are defined as follows, for any $E, F \in E(\mathcal{H})$:

$$E \oplus F := \begin{cases} E + F & \text{if} \quad E + F \in E(\mathcal{H}) \\ \mathbb{1} & \text{otherwise} \end{cases}$$

where $+$ is the usual operator-sum. We have

$$E^* := \mathbb{1} - E$$

The structure $\mathcal{E}(\mathcal{H}) := (E(\mathcal{H}), \oplus, *, \mathbb{1}, \mathbb{0})$ is a QMV-algebra, called *standard QMV-algebra* (Giuntini, 1995a). The structure $\mathcal{E}(\mathcal{H})$, however, is not an MV-algebra (Giuntini, 1996).

Example 2.2 (Standard MV-algebra). Let $[0,1] \subseteq \mathbb{R}$. For all $a,b \in [0,1]$, let

$$a \oplus b := \text{Min}\{a + b, 1\} \qquad (\textit{truncated sum})$$

and

$$a^* := 1 - a$$

The structure $\mathcal{M}_{[0,1]} = ([0,1], \oplus, *, 1, 0)$ is an MV-algebra, called *standard MV-algebra.*

Let $\mathcal{M}$ be a QMV-algebra. We can define the following relation:

$$a \preccurlyeq b \qquad \text{iff} \quad a = a \Cap b,$$

It turns out that the structure $(M, \preccurlyeq, *, \mathbf{1}, \mathbf{0})$ is an involutive bounded poset (i.e., a bounded poset with an order-reversing involution), which is not generally a lattice. Further, the following De Morgan-type laws hold:

$$(a \Cap b)^* = a \Cup b^*$$

$$(a \Cup b)^* = a^* \Cap b^*$$

If $\mathcal{M}$ is an MV-algebra, then the structure $(M, \preccurlyeq, *, \mathbf{1}, \mathbf{0})$ is a distributive de Morgan lattice (Chang, 1957), where $\forall a,b \in M$, the *inf* (*sup*) of a and b is $a \Cap b$ ($a \Cup b$). In the standard MV-algebra $\mathcal{M}_{[0,1]}$, the relation $\preccurlyeq$ coincides with the restriction to $[0, 1]$ of the usual order of $\mathbb{R}$. Consequently, $\mathcal{M}_{[0,1]}$ is *linear* (*totally ordered*): $\forall a, b \in [0,1]$: $a \preccurlyeq b$ or $b \preccurlyeq a$.

It turns out that $a \odot b = \text{Max}\{a + b - 1, 0\}$, $a \Cap b = \text{Min}\{a, b\}$, and $a \Cup b = \text{Max}\ \{a,b\}$.

Lemma 2.2 (Cancellation law). Let $\mathcal{M}$ be a QMV-algebra. For any $a, b, c \in M$: if $a \oplus c = b \oplus c$, $a \preccurlyeq c^*$, and $b \preccurlyeq c^*$, then $a = b$.

Lemma 2.3. Let $\mathcal{M}$ be a QMV-algebra. The following properties hold:
(i) If $a \preccurlyeq b$, then $b^* \preccurlyeq a^*$.
(ii) $a \preccurlyeq b$ iff $b = b \Cup a = a \Cup b$.
(iii) $a \Cap (b \Cup a) = a$.

Lemma 2.4. Let $\mathcal{M}$ be a QMV-algebra. The following properties hold:
(i) If $a \preccurlyeq b$, then $\forall c \in M$: $a \Cap c \preccurlyeq b \Cap c$ (weak monotony of $\Cap$).
(ii) If $a \preccurlyeq b$, then $\forall c \in M$: $a \Cup c \preccurlyeq b \Cup c$ (weak monotony of $\Cup$).
(iii) $a \Cap b \preccurlyeq b$ and $b \preccurlyeq a \Cup b$.

It should be noticed that, in general, $a \Cap b \not\preccurlyeq a$, $a \not\preccurlyeq a \Cup b$, and $a \Cap b \not\preccurlyeq b \Cup a$.

Lemma 2.5 (Monotony of $\oplus$ *and* $\odot$*).* Let $\mathcal{M}$ be a QMV-algebra. The following properties hold:
(i) If $a \preccurlyeq b$, then $\forall c \in M$: $a \oplus c \preccurlyeq b \oplus c$.
(ii) If $a \preccurlyeq b$, then $\forall c \in M$: $a \odot c \preccurlyeq b \odot c$.
(iii) If $a \preccurlyeq b$ and $c \preccurlyeq d$, then $a \oplus c \preccurlyeq b \oplus d$.
(iv) $a \preccurlyeq b$ and $c \preccurlyeq d$, then $a \odot c \preccurlyeq b \odot d$.

Lemma 2.6. Let $\mathcal{M}$ be a QMV-algebra. The following properties hold:
(i) $a \odot b \preccurlyeq a$.
(ii) $a \preccurlyeq a \oplus b$.
(iii) $a \odot b \preccurlyeq a \Cap b$, $a \odot b \preccurlyeq b \Cap a$.
(iv) $a \Cup b \preccurlyeq a \oplus b$, $b \Cup a \preccurlyeq a \oplus b$.

Definition 2.3. A *quasilinear QMV-algebra* is a QMV-algebra $\mathcal{M} = (M, \oplus, {}^*, \mathbf{1}, \mathbf{0})$ that satisfies the following condition $\forall a, b \in M$:

(QL) $a \not\preccurlyeq b \Rightarrow a \Cap b = b$

It is easy to see that the standard QMV-algebra $\mathcal{E}(H)$ of all effects on a Hilbert space $\mathcal{H}$ (see Example 2.1) is quasilinear.

Lemma 2.7. Let $\mathcal{M}$ be a QMV-algebra. The following conditions are equivalent:
(i) $\mathcal{M}$ is quasilinear.
(ii) $\forall a,b \in M$: $a \not\preccurlyeq b \Rightarrow a \Cap b = b$.
(iii) $\forall a,b,c \in M$: if $a \oplus c = b \oplus c \neq \mathbf{1}$, then $a = b$.

It should be noticed that any MV-algebra is quasilinear iff it is totally ordered.

3. THE CENTER OF A QMV-ALGEBRA

MV-algebras can be characterized as those QMV-algebras where the operation $\Cap$ is commutative. Therefore, it is quite natural to define a relation of commutativity for QMV-algebras, which turns out to be universal for MV-algebras. The notion of *center* can be defined in the usual manner. The main result of this section is that the center of a QMV-algebra $\mathcal{M}$ is an MV-subalgebra of $\mathcal{M}$ and therefore a distributive De Morgan lattice. We will freely use De Morgan laws and Lemmas 2.1–2.7 throughout this section.

Lemma 3.1. Let $\mathcal{M}$ be a QMV-algebra. Then, $\forall a, b \in M$

$$a = a \odot b^* \oplus (b \Cap a).$$

Proof. $a \odot b^* \oplus (b \Cap a) = a \odot b^* \oplus (b \oplus a^*) \odot a = a \odot b^* \oplus (b^* \odot a)^* \odot a = a \Cup (b^* \odot a) = a$, since $b^* \odot a \preccurlyeq a$. ■

Definition 3.1. Let $\mathcal{M}$ be a QMV-algebra and let $a, b \in M$. We say that *a commutes* with b ($a\mathbf{C}b$) iff $a \Cap b = b \Cap a$.

By definition, a QMV-algebra $\mathcal{M}$ is an MV-algebra iff $\forall a, b \in M$: $a\mathbf{C}b$.

Lemma 3.2. Let $\mathcal{M}$ be a QMV-algebra. The following conditions are equivalent $\forall$,$a,b \in M$:

(i) $a\mathbf{C}b$.

(ii) $a = a \odot b^* \oplus (a \Cap b)$.

Proof. (i) $\Rightarrow$ (ii) The proof follows from Lemma 3.1.

(ii) $\Rightarrow$ (i) Suppose $a = a \odot b^* \oplus (a \Cap b)$. By Lemma 3.1, $a = a \odot b^* \oplus (b \Cap a)$. Now, $a \odot b^* \preccurlyeq b \preccurlyeq a^* \odot b \oplus b^* = (a \Cap b)$ and $a \odot b^* \preccurlyeq b^* \odot a \oplus a^* = (b \Cap a)^*$. Thus, by the cancellation law (Lemma 2.2), $a \Cap b = b \Cap a$. ■

As proved in Giuntini (1996) every orthomodular lattice $\mathcal{L} = (L, \sqcap, \sqcup, ', \mathbf{1}, 0)$ (where $\sqcap$ ($\sqcup$) is the lattice-theoretic operations of *inf* (*sup*) and $'$ is the orthocomplement) can be thought of as a QMV-algebra, defining $\oplus = \sqcup$ and $* = '$. It turns out that $\forall a, b \in L$: $a\mathbf{C}b$ iff $a = (a \sqcap b^*) \sqcup (a \sqcap b)$ iff $\exists a_1, b_1, c \in L$ s.t. $a_1 = a_1 \sqcap c'$, $b_1 = b_1 \sqcap c'$, $a = a_1 \sqcup c$, and $b = b_1 \sqcup c$. Consequently, the relation $\mathbf{C}$ coincides with the usual notion of commutativity in orthomodular lattice theory (Kalnbach, 1983).

Lemma 3.3. Let $\mathcal{M}$ be a QMV-algebra. The following properties hold $\forall a, b \in M$:

(i) The relation $\mathbf{C}$ is reflexive and symmetric.

(ii) If $a \preccurlyeq b$ or $b \preccurlyeq a$, then $a\mathbf{C}b$.

(iii) If $a\mathbf{C}b$, then $d^*\mathbf{C}b^*$.

Proof. The proof of (i)–(ii) is straightforward.

(iii) Suppose $a\mathbf{C}b$; we have to prove $\grave{a}^* \Cap \acute{b}^* = \acute{b}^* \Cap \grave{a}^*$, or equivalently $a \Cup b = b \Cup a$. By Lemma 3.1, $b = b \odot a^* \oplus a \Cap b$. By hypothesis and Lemma

3.1, $a = a \odot b^* \oplus a \Cap b$. Thus, $a \Cup b = a \odot b^* \oplus b = a \odot b^* \oplus b \odot a^* \oplus a \Cap b = b \odot a^* \oplus a = b \Cup a$. ■

Lemma 3.4. Let $\mathcal{M}$ be a QMV algebra. Then, $\forall a, b, c \in M$: $(c \Cap a)^* \oplus (b \Cap a) = (c \Cap a)^* \oplus b$.

Proof:

$$
\begin{aligned}
(c \Cap a)^* \oplus (b \Cap a) &= (c^* \Cup a^*) \oplus (b \Cap a) \\
&= (c^* \odot a) \oplus a^* \oplus (b \Cap a) \\
&= (c^* \odot a) \oplus a^* \oplus b && \text{(QMV9)} \\
&= (c^* \Cup a^*) \oplus b \\
&= (c \Cap a)^* \oplus b
\end{aligned}
$$

Lemma 3.5. Let $\mathcal{M}$ be a QMV-algebra. $\forall a, b, c, d \in M$: if $a \preccurlyeq b$, $c \preccurlyeq d$, $b\mathsf{C}c$, and $b\mathsf{C}d$, then $a \Cup c \preccurlyeq b \Cup d$.

Proof. By Lemma 2.4(ii), $a \Cup c \preccurlyeq b \Cup c$ and $c \Cup b \preccurlyeq d \Cup b$. By Lemma 3.3(iii), $b^*\mathsf{C}d^*$ and $b^*\mathsf{C}d^*$. Hence: $b \Cup c = c \Cup b$ and $d \Cup b = b \Cup d$. Thus, $a \Cup c \preccurlyeq b \Cup d$. ■

Lemma 3.6. Let $\mathcal{M}$ be a QMV-algebra. $\forall a, b, c \in M$: if $a\mathsf{C}b$ and $a\mathsf{C}c$, then $(a \Cap b) \Cap c = b \Cap (a \Cap c)$.

Proof:

$$
\begin{aligned}
(a \Cap b) \Cap c &= (b \Cap a) \Cap c && (a\mathsf{C}b) \\
&= (b \Cap a) \Cap (a \Cap c) && \text{(QMV8)} \\
&= ((b \Cap a) \oplus (a \Cap c)^*) \odot (a \Cap c) \\
&= ((b \Cap a) \oplus (c \Cap a)^*) \odot (c \Cap a) && (a\mathsf{C}c) \\
&= ((c \Cap a)^* \oplus b) \odot (c \Cap a) && \text{(Lemma 3.4)} \\
&= b \Cap (c \Cap a) \\
&= b \Cap (a \Cap c) && (a\mathsf{C}c)
\end{aligned}
$$

Lemma 3.7. Let $\mathcal{M}$ a QMV-algebra. $\forall a, b, c \in M$: if $a\mathsf{C}b$, $a\mathsf{C}c$, and $b\mathsf{C}c$, then $(a \Cap b) \Cap c = c \Cap (a \Cap b)$.

Proof:

$$
\begin{aligned}
(a \Cap b) \Cap c &= b \Cap (a \Cap c) && \text{(Lemma 3.6)} \\
&= b \Cap (c \Cap a) && (a\mathsf{C}c)
\end{aligned}
$$

$$= (c \Cap b) \Cap a \qquad \text{(Lemma 3.6)}$$

$$= (b \Cap c) \Cap a \qquad (b\mathsf{C}c)$$

$$= c \Cap (b \Cap a) \qquad \text{(Lemma 3.6)}$$

$$= c \Cap (a \Cap b) \qquad (a\mathsf{C}b)$$

Lemma 3.8. Let $\mathcal{M}$ be a QMV-algebra. The following property holds $\forall a, b, c \in M$ s.t. $a\mathsf{C}c^*$ and $b\mathsf{C}c$:

$$a \oplus (b \Cap c) = (a \oplus b) \Cap (a \oplus c)$$

Proof. First, we prove $a \oplus (b \Cap c) \preccurlyeq (a \oplus b) \Cap (a \oplus c)$.

Let $\alpha := (a \oplus (b \Cap c)) \Cap ((a \oplus b) \Cap (a \oplus c))$. We want to show that $\alpha = a \oplus (b \Cap c)$:

$$\alpha = (a \oplus (b \Cap c)) \oplus ((a \oplus b)^* \Cup (a \oplus c)^*) \odot ((a \oplus b) \oplus (a \oplus c)^*) \odot (a \oplus c)$$

$$= (a \oplus (b \Cap c) \oplus (a \oplus b)^* \odot (a \oplus c) \oplus (a \oplus c)^*) \odot ((a \oplus b) \oplus (a \oplus c)^*) \odot (a \oplus c)$$

$$= ((a \oplus (b \Cap c) \oplus (a \oplus c)^*) \Cap ((a \oplus b) \oplus (a \oplus c)^*)) \odot (a \oplus c)$$

$$= ((a \oplus (c \Cap b) \oplus (a \oplus c)^*) \Cap ((a \oplus b) \oplus (a \oplus c)^*)) \odot (a \oplus c) \qquad (b\mathsf{C}c)$$

$$= (a \oplus (b \Cap c) \oplus (a \oplus c)^*) \odot (a \oplus c) \qquad \text{(Lemma 2.4(iii)–2.5(i))}$$

$$= (a \oplus (b \Cap c)) \Cap (a \oplus c)$$

$$= a \oplus (b \Cap c) \qquad \text{(Lemma 2.4(iii)–2.5(i))}$$

Since $a \oplus (b \Cap c) \preccurlyeq (a \oplus b) \Cap (a \oplus c)$, in order to prove $(a \oplus b) \Cap (a \oplus c) \preccurlyeq a \oplus (b \Cap c)$ it suffices to show

$$((a \oplus b) \Cap (a \oplus c))^* \oplus a \oplus (b \Cap c) = \mathbf{1}$$

Let $\beta := ((a \oplus b) \Cap (a \oplus c))^* \oplus a \oplus (b \Cap c)$. Then

$$\beta = a \oplus (b \Cap c) \oplus ((a \oplus b)^* \Cup (a \oplus c)^*)$$

$$= a \oplus (b \Cap c) \oplus (a^* \odot b^*) \odot (a \oplus c) \oplus (a^* \odot c^*)$$

$$= (c^* \Cup a) \oplus (b \Cap c) \oplus (\grave{a}^* \odot b^*) \odot (a \oplus c)$$

$$= (c^* \Cup a) \oplus (b \Cap c) \oplus b^* \odot (c \Cap a^*)$$

$$= b^* \odot (a^* \Cap c) \oplus (a^* \Cap c)^* \oplus (b \Cap c) \qquad (a\mathsf{C}c^*)$$

$$= b^* \odot (a^* \Cap c) \oplus (a^* \Cap c)^* \oplus b \qquad \text{(Lemma 3.4)}$$

$$= \mathbf{1}$$

Definition 3.2. Let $\mathcal{M}$ be a QMV-algebra. The *center* of $\mathcal{M}$ ($\mathfrak{Z}(M)$) is the set $\{a \in M \mid \forall b \in M: a\mathsf{C}b\}$.

Clearly, $\mathfrak{Z}(M) \neq \emptyset$, since $\mathbf{0}, \mathbf{1} \in \mathfrak{Z}(M)$.

Theorem 3.1. Let $\mathcal{M}$ be a QMV-algebra. $\forall a, b \in \mathfrak{Z}(M)$ s.t. $a \preccurlyeq b^*$: $a \oplus b \in \mathfrak{Z}(M)$.

Proof. Suppose $a, b \in \mathfrak{Z}(M)$. By Lemma 3.3(iii), we have $a^*, b^* \in \mathfrak{Z}(M)$. Let c be any element of M. By Lemma 3.1,

$$c = c \odot a^* \oplus (c \Cap a)$$

Again, by Lemma 3.1 and hypothesis

$$c \odot a^* = (c \odot a^*) \odot b^* \oplus ((c \odot a^*) \Cap b)$$

and

$$c \Cap a = (c \Cap a) \odot b \oplus ((c \Cap a) \Cap b^*)$$

Thus,

$$c = (c \odot a^*) \odot b^* \oplus ((c \odot a^*) \Cap b) \oplus (c \Cap a) \odot b \oplus ((c \Cap a) \Cap b^*)$$

By hypothesis, $a \preccurlyeq b^*$; hence: $a \odot b = \mathbf{0}$ and $a \Cap b^* = b^*$. Therefore, $(c \Cap a) \odot b = (c \oplus a^*) \odot a \odot b = \mathbf{0}.$ Further, $c \Cap a \preccurlyeq a \preccurlyeq b^*$, so that $(c \Cap a) \Cap b^* = c \Cap a$. Thus,

$$\begin{aligned}
c &= (c \odot a^*) \odot b^* \oplus ((c \odot a^*) \Cap b) \oplus (c \Cap a) \\
&= (c \odot a^*) \odot b^* \oplus (((c \Cap a) \oplus (c \odot a^*)) \Cap ((c \Cap a) \oplus b)) && \text{(Lemma 3.8)} \\
&= (c \odot a^*) \odot b^* \oplus (((a \Cap c) \oplus (c \odot a^*)) \Cap ((c \Cap a) \oplus b)) && (a\mathsf{C}c) \\
&= (c \odot a^*) \odot b^* \oplus (((a \oplus c^*) \odot c \oplus (c^* \oplus a)^*) \Cap ((c \Cap a) \oplus b)) \\
&= (c \odot a^*) \odot b^* \oplus ((c \Cup (c^* \oplus a)^*) \Cap ((c \Cap a) \oplus b)) \\
&= (c \odot a^*) \odot b^* \oplus (c \Cap ((c \Cap a) \oplus b)) && (c \odot a^* \preccurlyeq c) \\
&= (c \odot a^*) \odot b^* \oplus (c \Cap ((b \oplus c) \Cap (a \oplus b))) && \text{(Lemma 3.8)} \\
&= (c \odot a^*) \odot b^* \oplus (((b \oplus c) \Cap c) \Cap (a \oplus b)) && \text{(Lemma 3.6)} \\
&= (c \odot a^*) \odot b^* \oplus (c \Cap (a \oplus b)) \\
&= c \odot (a \oplus b)^* \oplus (c \Cap (a \oplus b))
\end{aligned}$$

Thus, by Lemma 3.2, $(a \oplus b)\mathsf{C}c$. ■

Theorem 3.2 (Main Theorem). Let $\mathcal{M}$ be a QMV-algebra. The structure $\mathfrak{Z}(\mathcal{M}) = (\mathfrak{Z}(M), \oplus, {}^*, \mathbf{1}, \mathbf{0})$ is an MV-subalgebra of $\mathcal{M}$. Further, $\forall a, b \in \mathfrak{Z}(M)$, the *inf* (*sup*) of a, b in $\mathfrak{Z}(\mathcal{M})$ coincides with the *inf* (*sup*) of a, b in

$\mathcal{M}$, which is equal to $a \Cap b$ ($a \Cup b$). Thus, ($\mathfrak{L}(M)$, $\Cap$, $\Cup$, *, **1**, **0**) is a De Morgan distributive sublattice of the involutive bounded poset (M, $\preccurlyeq$ *, **1**, **0**).

Proof. By Lemma 3.3(ii), $\mathfrak{L}(M)$ is closed under the operation * and contains **1, 0.** Thus, in order to prove that $\mathfrak{L}(\mathcal{M})$ is an MV-subalgebra of $\mathcal{M}$ it is sufficient to show that $\mathfrak{L}(M)$ is closed under $\oplus$. Suppose $a,b \in \mathfrak{L}(M)$. By (QMV9), we have $a \oplus b = a \oplus (b \Cap a^*)$. Since $\mathfrak{L}(M)$ is closed under $\Cap$ and * and $a \preccurlyeq b^* \Cup a = (b \Cap a^*)^*$, it follows by Theorem 3.1 that $a \oplus b \in \mathfrak{L}(M)$. Thus $\mathfrak{L}(\mathcal{M})$ is an MV-subalgebra of $\mathcal{M}$.

Let $a,b \in \mathfrak{L}(M)$. By commutativity, $a \Cap b \preccurlyeq a, b$. Let c be any element of $\mathcal{M}$ s.t. $c \preccurlyeq a, b$. By Lemma 2.4(iii), $c \preccurlyeq a \Cap b$. Thus, $a \Cap b$ is the inf of a, b in $\mathcal{M}$ and consequently in $\mathfrak{L}(M)$. By Chang (1957) we obtain that ($\mathfrak{L}(M)$, $\Cap$, $\Cup$, *, **1, 0**) is a De Morgan distributive sublattice of the involutive bounded poset (M, $\preccurlyeq$ *, **1**, **0**). ■

Corollary 3.1. Let $\mathcal{L}$ = (L, $\sqcap$, $\sqcup$, ′, **1**, **0**) be an orthomodular lattice. Let $\oplus := \sqcup$ and * = 1. The center of $\mathcal{L}$ (understood as a QMV-algebra) is a Boolean subalgebra of $\mathcal{L}$ and coincides with the center of $\mathcal{L}$ (understood as a lattice).

Corollary 3.2. Let $\mathcal{M}$ be a quasilinear QMV-algebra. The center of $\mathcal{M}$ is a totally ordered MV-algebra.

Theorem 3.3. Let $\mathscr{E}(\mathscr{H})$ be the standard QMV-algebra (see Example 2.1). The center of $\mathscr{E}(\mathscr{H})$ is irreducible, i.e., $\mathfrak{L}(\mathscr{E}(\mathscr{H})) = \{\mathbb{0}, \mathbb{1}\}$.

Proof. Since $\mathscr{E}(\mathscr{H})$ is quasilinear, by Corollary 3.1, $\mathfrak{L}(\mathscr{E}(\mathscr{H}))$ is a totally ordered MV-subalgebra of $\mathscr{E}(\mathscr{H})$. Suppose $E \in E(\mathscr{H})$. We want to show that $E \in \{\mathbb{0}, \mathbb{1}\}$. Let P be any projection operator in $E(\mathscr{H})$. Two cases are possible: (i) $E \preccurlyeq P$; (ii) $P \preccurlyeq E$.

(i) Let $E \preccurlyeq P$. By Giuntini and Greuling (1989), we have $E = EP = PE$. If $E \preccurlyeq P^*$, then, again by Giuntini and Greuling and Greuling (1989), $E = EP^* = P^*E$. Hence: $E = EP = EP^*\ P = \mathbb{0}$ If $E \not\preccurlyeq P^*$, then $P^* \preccurlyeq E$ and therefore, by transitivity, $P^* \preccurlyeq P$. Hence: $P = \mathbb{1}$.

The proof of case (ii) is similar to the proof of case (i).

REFERENCES

Chang, C. C. (1957). Algebraic analysis of many valued logics, *Transactions of the American Mathematical Society,* **88**, 74–80.

Chang, C. C. (1958). A new proof of the completeness of Lukasiewicz axioms, *Transactions of the American Mathematical Society,* **93**, 467–490.

Giuntini, R. (1995a). Quasilinear QMV algebras, *International Journal of Theoretical Physics,* **34**, 1397–1407.

Giuntini, R. (1995b). Unsharp orthoalgebras and quantum MV algebras, in The Foundations of Quantum Mechanics—Historical Analysis and Open Questions, C. Garola and A. Rossi, eds., Kluwer, Dordrecht, pp. 325–337.

Giuntini, R. (1996). Quantum MV algebras, *Studia Logica,* **56**, 393–417.

Giuntini, R., and Greuling, H. (1989). Toward an unsharp language for unsharp properties, *Foundations of Physics,* **19**, 931–945.

Kalmbach, G. (1983). *Orthomodular Lattices*, Academic Press, New York.